WESLEY COLLEGE

By daring & by doing

YEAR 12 MATHEMATICS SPECIALIST

SEMESTER TWO 2019

TEST 6: Sample Means and Simple Harmonic Motion

Name: ANSWERS

Friday 20th September 2019

Time: 60 minutes

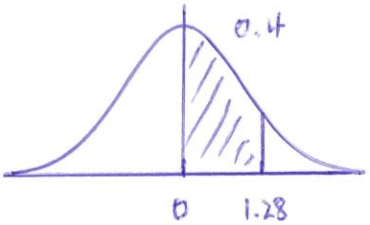
Total marks: $\frac{\quad}{10}+\frac{\quad}{40}=\frac{\quad}{50}$

Calculator free section – maximum 15 minutes

1. [4 marks – 2 and 2]

For any standard normal score z, $P(0 < z < 1.28) = 0.4$

(a) Set up an 80% confidence interval for a single score selected at random from a normally distributed population with mean 50 and standard deviation 10.



$50 \pm 1.28 \times 10$

$= [37.2, 62.8]$

(b) Set up an 80% confidence interval for the mean of 64 scores selected randomly and independently from a symmetric population with mean 50 and standard deviation 10

$50 \pm 1.28 \times \frac{10}{\sqrt{64}}$

$\frac{12.8}{8} = 1.6$

$= 50 \pm 1.6$

$= [48.4, 51.6]$

2. [6 marks – 3 and 3]

A particle is travelling in a straight line with velocity v related to displacement x by the equation: $v = \sqrt{12 - 4x^2}$

(a) Show that the particle is in simple harmonic motion.

$$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(\frac{1}{2}\left[12 - 4x^2\right]\right) = \frac{1}{2}x - 8x = -4x$$

OR $$a = v.\frac{dv}{dx} = \sqrt{12-4x^2} \times \frac{-8x}{2\sqrt{12-4x^2}} = -4x$$

OR $$a = \frac{dv}{dt} = \frac{-8x}{2\sqrt{12-4x^2}} \times \frac{dx}{dt} = -4x$$

then $a = -4x$

$= -2^2 x$ which is of the form $a = -k^2 x$ with $k = 2$

$\therefore$ SHM

or $v^2 = 12 - 4x^2 = 4(3 - x^2) = k^2(A^2 - x^2)$ and you have a very easy 6 marks!

(b) Determine the period and amplitude.

$$T = \frac{2\pi}{k} = \pi \text{ units}$$

Either $v = 0$ when $x = \pm A$

$\Rightarrow 12 - 4x^2 = 0$

$\Rightarrow x = \pm\sqrt{3}$

$\Rightarrow A = \sqrt{3}$

OR $v^2 = k^2(A^2 - x^2)$

$\therefore 12 - 4x^2 = 4(A^2 - x^2)$

$\Rightarrow 4A^2 = 12$

$\Rightarrow A = \sqrt{3}$ $(A > 0)$

Note: any SHM $x = A\sin(kt + \alpha)$ has $v = \cancel{\pm}\, kA\cos(kt + \alpha)$

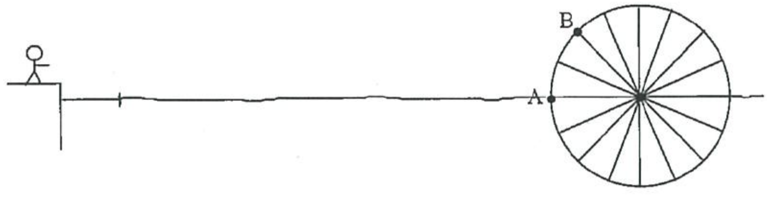
$= k\sqrt{A^2 - x^2}$

Year 12 Specialist Test 6: Calculator assumed section

Time: 45 minutes 40 marks Name: ____________________

3. [8 marks – 2, ~~1~~ 2, 1, ~~2~~ 1, and 2]

A Mathematics Specialist student is standing on a jetty (or wharf) and watching a paddle steamer in the distance. The vertical movement of the paddles is simple harmonic. Paddles *A* and *B* start as indicated.



The wheel has diameter 5 metres and is turning clockwise at one revolution every 8 seconds. *A* is initially at the water level.

(a) Write an equation for the height h_A of paddle *A* above the water level at time *t* seconds.

$T = 8 = \frac{2\pi}{k}$

$k = \frac{\pi}{4}$ ✓

$h(t) = 2.5 \sin \frac{\pi t}{4}$ ✓

$\left(\text{or } 2.5 \cos\left[\frac{\pi t}{4} - \frac{\pi}{2}\right]\right)$

(b) Give an equation for the height h_B of paddle *B* above the water level at time *t* seconds.

$h(t) = 2.5 \sin\left(\frac{\pi}{4}t + \frac{\pi}{4}\right)$ ✓✓

$= 2.5 \sin \frac{\pi}{4}(t+1)$

or $2.5 \cos \frac{\pi}{4}(t-1)$

(c) Write an equation for the acceleration of *A*, $\frac{d^2h_A}{dt^2}$, in terms of h_A.

$\frac{d^2h_A}{dt^2} = -\frac{\pi^2}{16} h_A$ ✓

(d) When does h_A first equal h_B?

$t = 1.5$ seconds. ✓

(Solve $h_A = h_B$ or place A in 3^{rd} pos^{n}, B in the 5^{th})

(e) What is the vertical velocity of *A*, $\frac{dh_A}{dt}$, when $h_A = 1.5$ metres?

$h_A(t) = 1.5 \Rightarrow t = 0.819$ ✓

$v(0.819) = 1.57$ ✓

or

$v^2 = k^2(A^2 - x^2)$

$= \frac{\pi^2}{16}(2.5^2 - 1.5^2)$ ✓

$\Rightarrow v = \frac{\pi}{4} \times 2 = \frac{\pi}{2}$ ✓

4. [11 marks – 3, 3, 2 and 3]

The probability that anyone will recall an advertisement shown in a cinema is 0.58.

100 people were questioned as they left the Cygnet Theatre after seeing "Downton Abbey" and the number who recalled a particular advertisement was recorded as a random variable x.

(a) Describe the probability distribution for x and specify its mean and standard deviation

$x \sim$ Binomial $\quad n = 100 \quad p = 0.58$ ✓

$\mu = np = 58$ ✓

$\sigma = \sqrt{np(1-p)} = 4.9356$ ✓

This process was repeated on 40 separate occasions and the average number of people recalling the advertisement was calculated as a random variable X.

(b) Describe and justify the probability distribution for X and specify its mean and standard deviation

$X \sim$ normal ✓ by C.L.T. ✓ (Central limit theorem) as $n > 30$

$\bar{X} = 58$

$S_{\bar{x}} = \dfrac{4.9356}{\sqrt{40}} = 0.7804$ ✓

(c) What is the probability the observed average was greater than 59?

$P(X > 59) =$ normCDf$(59, \infty, 0.7804, 58)$ ✓

$= 0.100$ ✓

(d) Assuming the same sample mean of 58, how could you ensure that a 95% confidence interval for the population mean is $57 < \mu < 59$?

$\bar{X} \pm \dfrac{Z S_x}{\sqrt{n}} \Rightarrow \dfrac{1.96 \times 4.9356}{\sqrt{n}}$ ✓ $= 1$ ✓ $\Rightarrow n = 94$ ✓

i.e. ~~a sample size of 94~~ (or greater)

94 Samples

5. [10 marks – 1, 2, 1, 3, 2 and 1]

A predator-prey relationship, such lions and antelope, has the rate of growth of the predator population, x, directly proportional to the prey population, y, and the rate of <u>decline</u> of the prey population directly proportional to the predator population.

$$\text{i.e. } \frac{dx}{dt} = ay \text{ and } \frac{dy}{dt} = -bx \quad \text{for constants } a, b \text{ both} >0$$

(a) Express $\frac{d^2x}{dt^2}$ in terms of a, b and x

$$\frac{d^2x}{dt^2} = a.\frac{dy}{dt} = -abx \checkmark$$

(b) Show that $x(t)$ is a simple harmonic relationship

$$\frac{d^2x}{dt^2} = -k^2x \text{ with } \checkmark\ k^2 = ab$$

$\Rightarrow$ SHM $\checkmark$

When $a = 0.01$ and $b = 1$, and for an initial predator population of 200 and initial prey population of 4000, find:

(c) $\frac{dx}{dt}$ at $t = 0$

$$\frac{dx}{dt} = ay = 40 \checkmark$$

(d) k and A if $x(t) = A\sin(kt + \alpha)$

$$k = \sqrt{ab} = 0.1 \checkmark$$

$v^2 = k^2(A^2 - x^2)$ $\checkmark$

$1600 = 0.01(A^2 - 40000)$

$\Rightarrow A = 447.2$ $\checkmark$

$(200\sqrt{5})$

or $A\sin\alpha = 200$

$Ak\cos\alpha = 40$ $\checkmark$

$A\cos\alpha = 400$

$\alpha = 0.46365$; $A = 447.2$ $\checkmark$

(from $\tan\alpha = 0.5$)

(e) $x(t)$

$= 447.2\sin(0.1t + \alpha)$

$\alpha = 0.46365$

$$x(t) = 447.2\sin(0.1t + 0.46365) \checkmark \checkmark$$

(f) When the predator species becomes extinct.

$$x(t) = 0 \Rightarrow t = \frac{\pi - \alpha}{0.1} = 26.78 \text{ yrs.} \checkmark$$

6. [11 marks – 2, 3, 2, 3 and 2]

Ambulance ramping is the practice of leaving a patient in the parked ambulance outside a hospital, rather than an immediate transfer to admissions or ED.

A survey last Monday of the 40 ambulances ramped outside Perth hospitals had an average ramping time of 97 minutes, with a standard deviation of 32 minutes.

(a) Describe the distribution from which this mean of 97 minutes is drawn.

Normal $\mu = 97$ ✓, $\sigma = \frac{32}{\sqrt{40}} \approx 5.06$ ✓

(b) Justify your choices in (a) and list any assumptions made.

Central Limit Theorem justifies these choices.
Assume a random sample
Assume μ & σ are close to population statistics
Assume n large enough
etc ...

3 pts ✓

(c) Establish a 95% confidence interval for the population mean μ of average ambulance ramping times.

$97 \pm 1.96 \times \frac{32}{\sqrt{40}}$ ✓ is $[87.1, 106.9]$ ✓

(d) A 99% confidence interval of [70.0, 87.4] for the population mean of average ramping times was developed from another sample of size 40 by the Health Minister's task force. Determine the new sample mean and new sample standard deviation.

$\mu = \frac{157.4}{2} = 78.7$ ✓

$78.7 \pm d \Rightarrow d = 8.7$ ✓ $= \frac{z\, s_x}{\sqrt{n}} = \frac{2.576 \times s_x}{\sqrt{40}}$

$s_x = 21.36$ ✓

(e) Within which of the two intervals is the population mean more likely to occur? Explain your choice.

✓ Could fall in either, could fall in both (they overlap) or neither
Higher confidence suggests mean more likely to be
✓ in [70.4, 87.4], even though it is slightly narrower.